Appendix

A Proofs of Propositions

Proposition 1. *Let n be an odd integer such that $n \geq 5$. Then, $\eta(C_n) = 3$.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ and suppose that $f : V \to \{1, 2\}$ is an additive 2-coloring of C_n. Then, f is also a topological additive 2-numbering of a certain digraph D such that $G(D) = C_n$. Observe that $f(N(v)) \in \{2, 3, 4\}$ for all $v \in V$. Since C_n is not bipartite, there must be an oriented path of 3 consecutive vertices in D. W.l.o.g. assume that $f(N(v_2)) < f(N(v_3)) < f(N(v_4))$. Then, $f(v_1) + f(v_3) = f(N(v_2)) = 2$ and we obtain $f(v_3) = 1$. But, $f(v_3) + f(v_5) = f(N(v_4)) = 4$ giving $f(v_5) = 3$ which is an absurd. Therefore, $\eta(C_n) \geq 3$.

Consider the assignment $f : V \to [3]$ such that $f(v_2) = f(v_4) = f(v_5) = 1$, $f(v_1) = 2$, $f(v_3) = 3$ and, if $n \geq 7$, then for $i \geq 6$, $f(v_i) = 1$ if i is even and $f(v_i) = 3$ if i is odd. We obtain $f(N(v_1)) = 2$ if $n = 5$ and $f(N(v_1)) = 4$ otherwise, $f(N(v_2)) = 5$, $f(N(v_3)) = 2$, $f(N(v_4)) = 4$ and $f(N(v_n)) = 3$. If $n \geq 7$, $f(N(v_5)) = 2$ and $f(N(v_6)) = 4$. If $n \geq 9$, then for $i \in \{7, \ldots, n-1\}$, $f(N(v_i)) = 2$ if i is odd and $f(N(v_i)) = 6$ if i is even. Thus, f is an additive 3-coloring of C_n. □

Proposition 2. *Let $G = (Q' \cup S', E)$ be a complete split. Then, $\eta(G) = |Q'|$.*

Proof. Since G has $|Q'|$ true twins, $\eta(G) \geq |Q'|$. On the other hand, let $v \in S'$ and $Q = Q' \cup \{v\}$. Here, Q is a maximal clique of G. By applying Theorem 3 with $T = \{u, v\}$ where $u \in Q'$, we obtain $\eta(G) = |Q| - 1 = |Q'|$. □

Proposition 3. *Let G be a thin/thick headless spider of order q. Then,*

$$\eta(G) = \left\lceil \frac{q+1}{2} \right\rceil.$$

Proof. For the sake of simplicity, we call $r = \lceil \frac{q+1}{2} \rceil$. We start by proving $\eta(G) = r$ when G is thin. Note that $d(u_i) = q$ for all i. In virtue of Corollary we have $\eta(G) \geq r$. Then, we only need to propose an additive r-coloring of G. If $q = 2$, consider the additive 2-coloring f such that $f(u_1) = f(u_2) = f(v_1) = 1$ and $f(v_2) = 2$. If $q \geq 3$, consider the assignment $f : V \to [r]$ such that $f(u_i) = r - i + 1$ and $f(v_i) = 1$ for all $i \in [r]$, and

$f(u_i) = q - i + 1$ and $f(v_i) = \lfloor \frac{q+1}{2} \rfloor$ for all $i \in \{r+1, \ldots, q\}$. We obtain $f(N(u_i)) = f(Q) - f(u_i) + f(v_i) = f(Q) - r + i$ for all $i \in [q]$. Then, for $j < k$, we have $f(N(u_j)) < f(N(u_k))$. Regarding the edge (u_i, v_i), we first analyze when $i = 1$. Note that $f(u_1) = r$, $f(u_2) = r - 1$ and $f(u_q) = 1$, then $f(N(u_1)) = f(Q) - r + 1 \geq f(u_1) + f(u_2) + f(u_q) - r + 1 = r + 1 > r = f(N(v_1))$. If $i \geq 2$, $f(N(u_i)) > f(N(u_1)) > f(N(v_1)) = r \geq f(u_i) = f(N(v_i))$.

Now, we consider that G is thick. If $q = 2$ then G is isomorphic to a thin headless spider of order 2. Hence, assume that $q \geq 3$. Consider the assignment $f : V \to [r]$ such that $f(u_i) = i$ and $f(v_i) = 1$ for all $i \in [r]$, and $f(u_i) = r$ and $f(v_i) = i - r + 1$ for all $i \in \{r+1, \ldots, q\}$. We obtain $f(N(u_i)) = f(V) - f(u_i) - f(v_i) = f(V) - i - 1$ for all $i \in [q]$. Then, for $j < k$, we have $f(N(u_j)) > f(N(u_k))$. Regarding the edge (u_j, v_k), $j \neq k$, we prove that $f(N(v_j)) \leq f(N(v_1)) < f(N(u_q)) \leq f(N(u_k))$. As $q \geq k$, the right inequality holds. The left inequality holds since $f(N(v_j)) = f(Q) - f(u_j) \leq f(Q) - 1 = f(N(v_1))$. The middle inequality, i.e. $f(N(v_1)) < f(N(u_q))$, holds if and only if $f(V) - f(Q) > q$. Observe that

$$f(V) - f(Q) = \sum_{i=1}^{q} f(v_i) = r + \sum_{i=r+1}^{q} (i + 1 - r) = q + \frac{(q-r)^2 + q - r}{2} > q.$$

We finish by proving that $\eta(G) \geq r$. Suppose that there exists an additive $(r-1)$-coloring f of G. Recall that $f(N(u_i)) = f(V) - f(u_i) - f(v_i)$ for all $i \in [q]$. Thus, $f(V) - (2r-2) \leq f(N(u_i)) \leq f(V) - 2$. Since there are $2r - 3$ integers in the range of feasible values for $f(N(u_i))$ and $2r - 3 < q$, there are two indexes j and k such that $f(N(u_j)) = f(N(u_k))$ by the pigeonhole principle, leading to a contradiction. □

Proposition 4. *Let $m \geq 3$. Then, $\eta(KS_m) = \lceil \frac{m+2}{3} \rceil$.*

Proof. For the sake of simplicity, we call $r = \lceil \frac{m+2}{3} \rceil$. Note that $d(u_i) = m + 1$ for all i. In virtue of Corollary 2 we have $\eta(G) \geq r$.

We only have to propose an additive r-coloring of KS_m. First, define a permutation function $\mathfrak{p} : [m] \to [m]$ as follows: $\mathfrak{p}(1) = 1$, $\mathfrak{p}(j) = \frac{j}{2} + 1$ for $j = 2, \ldots, m$ and j even, $\mathfrak{p}(j) = m - \frac{j-3}{2}$ for $j = 3, \ldots, m$ and j odd. Clearly, its inverse is: $\mathfrak{q}(1) = 1$, $\mathfrak{q}(i) = 2(i-1)$ for $i = 2, \ldots, \lfloor \frac{m}{2} \rfloor + 1$, $\mathfrak{q}(i) = 3 + 2(m-i)$ for $i = \lfloor \frac{m}{2} \rfloor + 2, \ldots, m$. Let f be the following assignment:

$$f(u_i) = \begin{cases} r, & m \equiv 2 \pmod 3 \ \wedge \ i = \mathfrak{p}(m), \\ \left\lfloor \dfrac{\mathfrak{q}(i)}{3} \right\rfloor + 1, & \text{otherwise.} \end{cases}$$

$$f(v_i) = \begin{cases} r + 1 - \left\lceil \dfrac{\mathfrak{q}(i)}{3} \right\rceil, & i = 1 \ \vee \ i \geq \left\lfloor \dfrac{m}{2} \right\rfloor + 2, \\ 2, & m \equiv 2 \pmod 6 \ \wedge \ i = \mathfrak{p}(m), \\ r + 1 - \left\lceil \dfrac{\mathfrak{q}(i) + 2}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

It is easy to check that $f(w) \in [r]$ for all $w \in U \cup V$. Also, observe that first and second case in the definition of $f(v_i)$ do not overlap: if $m \equiv 2 \pmod 6$, m is even and, therefore, $2 \leq \mathfrak{p}(m) = m/2 + 1 < \lfloor \frac{m}{2} \rfloor + 2$.

We claim that $f(v_i)$ satisfies the following recursive relationship:

$$f(v_i) = 2r - \mathfrak{q}(i) + f(u_i) - f(v_{i-1}), \quad \forall\, i \in [m].$$

Then, $f(N(u_i)) = f(U) - f(u_i) + f(v_i) + f(v_{i-1}) = f(U) + 2r - \mathfrak{q}(i)$ for all i. Since $\mathfrak{q}$ is injective, $f(N(u_i)) \neq f(N(u_k))$ for all $i \neq k$. Regarding edges between U and V, note that $f(U) > m$ and for any $v \in V$, v has degree 2, then $f(N(v)) \leq 2r < f(U) + 2r - m \leq f(U) + 2r - \mathfrak{q}(i) = f(N(u_i))$ for all i. Therefore, f is an additive r-coloring of KS_m.

Now, we check our claim. If $i \neq \mathfrak{p}(m) = \lceil \frac{m}{2} \rceil + 1$ or $m \not\equiv 2 \pmod 3$, then $f(u_i) - \mathfrak{q}(i) = 1 - \lceil \frac{2\mathfrak{q}(i)}{3} \rceil$. That is, we have to check $f(v_i) = 2r + 1 - \lceil \frac{2\mathfrak{q}(i)}{3} \rceil - f(v_{i-1})$. In the case that $m \equiv 2 \pmod 3$ and $i = \mathfrak{p}(m) = \lceil \frac{m}{2} \rceil + 1$, $f(u_i) - \mathfrak{q}(i) = r - m$ and we have to check $f(v_i) = 3r - m - f(v_{i-1})$.

1. Case $i = 1$: Since $f(v_0) = f(v_m) = r$, $f(v_1) = r + 1 - \lceil \frac{1}{3} \rceil = 2r + 1 - \lceil \frac{2}{3} \rceil - r$.

2. Case $i = 2$: Since $f(v_1) = r$, $f(v_2) = r + 1 - \lceil \frac{4}{3} \rceil = 2r + 1 - \lceil \frac{4}{3} \rceil - r$.

3. Case $i = 3, \ldots, \lfloor \frac{m}{2} \rfloor$ or "$i = \lfloor \frac{m}{2} \rfloor + 1$ when $m \not\equiv 2 \pmod 6$": First, we prove $1 - \lceil \frac{2(i-1)+2}{3} \rceil = \lceil \frac{2(i-1)}{3} \rceil - \lceil \frac{4(i-1)}{3} \rceil$. If $i \equiv 1 \pmod 3$, let $h = \frac{i-1}{3}$. Then, $1 - \lceil \frac{2(i-1)+2}{3} \rceil = 1 - 2h - \lceil \frac{2}{3} \rceil = 2h - 4h = \lceil \frac{2(i-1)}{3} \rceil - \lceil \frac{4(i-1)}{3} \rceil$. Cases when $i \equiv 0$ or $2 \pmod 3$ are analogous. Since $f(v_{i-1}) = r + 1 - \lceil \frac{2(i-1)}{3} \rceil$, $f(v_i) = r + 1 - \lceil \frac{2(i-1)+2}{3} \rceil = 2r + 1 - \lceil \frac{4(i-1)}{3} \rceil - r - 1 + \lceil \frac{2(i-1)}{3} \rceil$.

4. Case $i = \lfloor \frac{m}{2} \rfloor + 1$ when $m \equiv 2 \pmod 6$: Then, $r = \lceil \frac{m+2}{3} \rceil = \frac{m}{3} + 1$, $\mathfrak{q}(i) = m$, $\mathfrak{q}(i-1) = m - 2$, $f(v_{i-1}) = r + 1 - \lceil \frac{m-2+2}{3} \rceil = 1$ and $f(v_i) = 2 = 3r - m - 1$.

5. Case $i = \lfloor \frac{m}{2} \rfloor + 2$: If m is even, $\mathfrak{q}(i) = m - 1$ and $\mathfrak{q}(i-1) = m$. If $m \not\equiv 2 \pmod 3$, $f(v_{i-1}) = r + 1 - \lceil \frac{m+2}{3} \rceil = 1$. Note that $2r - \lceil \frac{2(m-1)}{3} \rceil = 2$.

If $m \equiv 2 \pmod 3$, then $m \equiv 2 \pmod 6$ and, therefore, $f(v_{i-1}) = 2$ and $2r - \lceil \frac{2(m-1)}{3} \rceil = 3$. Then, $f(v_i) = r + 1 - \lceil \frac{m-1}{3} \rceil = 2 = 2r + 1 - \lceil \frac{2(m-1)}{3} \rceil - f(v_{i-1})$; If m is odd, $\mathfrak{q}(i) = m$ and $\mathfrak{q}(i-1) = m-1$, If $m \not\equiv 2 \pmod 3$, note that $1 - \lceil \frac{m}{3} \rceil = \lceil \frac{m+1}{3} \rceil - \lceil \frac{2m}{3} \rceil$. Then, $f(v_i) = r + 1 - \lceil \frac{m}{3} \rceil = 2r + 1 - \lceil \frac{2m}{3} \rceil - r - 1 + \lceil \frac{m-1+2}{3} \rceil$. If $m \equiv 2 \pmod 3$, $r - 1 = \lceil \frac{m+2}{3} \rceil - 1 = \lceil \frac{m+1}{3} \rceil = \lceil \frac{m}{3} \rceil$ and $f(v_{i-1}) = r + 1 - \lceil \frac{m-1+2}{3} \rceil = 2$. Then, $f(v_i) = r + 1 - \lceil \frac{m}{3} \rceil = 2 = 3r - m - f(v_{i-1})$.

6. Case $i = \lfloor \frac{m}{2} \rfloor + 3, \ldots, m-1$: We have $f(v_i) = r + 1 - \lceil \frac{3+2(m-i)}{3} \rceil$ and $2r + 1 - \lceil \frac{2\mathfrak{q}(i)}{3} \rceil - f(v_{i-1}) = r - \lceil \frac{6+4(m-i)}{3} \rceil + \lceil \frac{3+2(m-i+1)}{3} \rceil$. To prove that both expressions are equal, we proceed as in the third case.

□

Proposition 5. *Let $m \geq 4$. Then, $\eta(CS_m) = \eta(WS_m) = 2$.*

Proof. By Observation 1 $\eta(CS_m) \geq 2$ and $\eta(WS_m) \geq 2$ so we only have to propose an additive 2-coloring of CS_m and WS_m. We start with CS_m.

Consider an assignment $f : V \to \{1, 2\}$ such that $f(u_i) = 2$ if i is odd, $f(u_i) = 1$ if i is even and $f(v) = 1$ for all $v \in V \backslash \{v_1\}$. If m is even, also assign $f(v_1) = 1$. Thus, $f(N(u_i)) = 4$ if i is odd, $f(N(u_i)) = 6$ if i is even and $f(N(v)) = 3$ for all $v \in V$. If m is odd, assign $f(v_1) = 2$. In this case, $f(N(u_1)) = 6$, $f(N(u_2)) = 7$, $f(N(u_m)) = 5$ and for $i = 3, \ldots, m-1$, $f(N(u_i)) = 4$ if i is odd and $f(N(u_i)) = 6$ if i is even. In addition, $f(N(v_m)) = 4$ and $f(N(v)) = 3$ for all $v \in V \backslash \{v_m\}$. Therefore, f is an additive 2-coloring of CS_m.

For WS_m, assume that $m \neq 5$ and consider the same assignment as before plus $f(w) = 1$. Then, values of $f(N(v))$ remains the same as in CS_m, values of $f(N(u))$ are the same as in CS_m plus one, i.e. $f(N_{WS_m}(u)) = f(N_{CS_m}(u)) + 1$, and $f(N(w)) = \lceil 3m/2 \rceil$. If $m = 4$, clearly f is an additive 2-coloring of WS_4. If $m \geq 6$, $f(N(w)) > 8 \geq f(N(u))$ and f is an additive 2-coloring of WS_m.

For $m = 5$, we propose a different additive 2-coloring of WS_5: $f(u_1) = f(u_2) = f(u_4) = f(v_4) = f(v_5) = 1$, $f(u_3) = f(u_5) = f(v_1) = f(v_2) = f(v_3) = f(w) = 2$. Then, $f(N(v_1)) = 2$, $f(N(v_i)) = 3$ for $i \in \{2, \ldots, 5\}$, $f(N(u_5)) = 6$, $f(N(w)) = 7$, $f(N(u_1)) = f(N(u_3)) = 8$ and $f(N(u_2)) = f(N(u_4)) = 9$. □

B Notes on the tool for solving the ACP

Let $G = (V, E)$ be a graph, $E_2 = \{(u, v), (v, u) : (u, v) \in E\}$ (edges occur in both directions), integer variables k and $f(v)$ for all $v \in V$, and binary variables $z(u, v)$ for all $(u, v) \in E_2$, where $z(u, v) = 1$ if and only if $f(N(u)) < f(N(v))$. The following integer programming formulation $\mathscr{F}$ computes $\eta(G)$:

$$
\begin{aligned}
&\min k \\
&\text{subject to} \\
&\qquad f(N(u)) - f(N(v)) + M_{uv} z(u, v) \leq M_{uv} - 1, && \forall\ (u, v) \in E_2 \\
&\qquad z(u, v) + z(v, u) = 1, && \forall\ (u, v) \in E \\
&\qquad 1 \leq f(v) \leq UB, && \forall\ v \in V \\
&\qquad f(v) \leq k, && \forall\ v \in V \\
&\qquad z(u, v) \in \{0, 1\}, && \forall\ (u, v) \in E_2 \\
&\qquad k, f(v) \in \mathbb{Z}_+, && \forall\ v \in V
\end{aligned}
$$

where $M_{uv} = 1 + |N(u) \backslash N(v)|UB - |N(v) \backslash N(u)|$ for all $(u, v) \in E_2$ and UB is an upper bound of $\eta(G)$.

Additional inequalities can be considered in order to improve the performance of the optimization. In particular, the initial relaxation of $\mathscr{F}$ can be reinforced by adding these valid inequalities:

$$z(v, w) + z(w, u) \leq 1, \quad \text{for all } u, v, w \text{ such that } (u, v) \notin E_2,\ w \in N(u) \subset N(v).$$

Note that if $z(v, w) = z(w, u) = 1$ then $f(N(v)) < f(N(w)) < f(N(u))$, which leads to a contradiction. We call $\mathscr{F}_1$ to the formulation with these inequalities.

On the other hand, symmetrical solutions arising from the presence of twin vertices can be partially removed with the following procedure. Let $\mathscr{C}$ be a partition of V, where each element of $\mathscr{C}$ can be: 1) a single vertex, 2) two or more false twins each other, and 3) two or more true twins each other. Then, for every set of false twins $\{v_1, \ldots, v_t\} \in \mathscr{C}$ add inequalities $f(v_i) \leq f(v_{i+1}),\ \ \forall\ i \in [t-1]$ and remove variables $z(u, v_i)$, $z(v_i, u)$ and constraints where they occur for all $i \in 2, \ldots, t$ and $u \in N(v_1)$. Analogously, for every set of true twins $\{v_1, \ldots, v_t\} \in \mathscr{C}$ add inequalities $f(v_i) \leq f(v_{i+1}) - 1,\ \ \forall\ i \in [t-1]$ and remove variables $z(v_i, v_j)$ and constraints where they occur for all $i, j = 2, \ldots, t$ such that $i \neq j$. We call $\mathscr{F}_2$ to the resulting formulation after applying this procedure to $\mathscr{F}_1$.

A suitable partition $\mathscr{C}$ can be generated as follows. First, compute a partition $\mathscr{C}'$ of V into maximal sets of true twins. Let $\mathscr{C}_1 \subset \mathscr{C}'$ composed only of singleton sets and $V' = \bigcup_{W \in \mathscr{C}_1} W$ (i.e. $V' = \{v \in V : \{v\} \in \mathscr{C}'\}$). Then, compute a partition $\mathscr{C}''$ of V' into maximal sets of false twins. Finally, do $\mathscr{C} \leftarrow (\mathscr{C}' \backslash \mathscr{C}_1) \cup \mathscr{C}''$.

We have run an experiment in order to know the size of graphs where ACP can be solved with our approach. A computer with an Intel i5 CPU 750@2.67GHz, Visual Studio 2013 and IBM ILOG CPLEX 12.6 has been used for the experiment. For each instance, the upper bound given by Akbari et al. (i.e. $UB = \Delta^2 - \Delta + 1$) [2] is computed. A limit of two hours is imposed to the optimization. In Table 1, we show the time in seconds needed to solve 27 random instances (3 per vertices-density combination) with $\mathscr{F}_1$. These instances were generated by starting from the empty graph of n vertices and adding edges with probability p, where $n \in \{20, 25, 30\}$ and $p \in \{0.25, 0.5, 0.75\}$ (low, medium and high density, respectively). A mark "$-$" means that the instance could not be solved in the term of two hours. The last three rows display results over instances of 50 vertices generated by adding 20 true, 20 false or a mix of 10 true and false twins to the random instances of 30 vertices with medium density. Time is reported in the form $\alpha(\beta)$ where α is the time consumed by $\mathscr{F}_2$ (including the procedure to generate partition $\mathscr{C}$) and β the time consumed by $\mathscr{F}_1$.

As we can see, the tool is able to solve almost all instances of 30 vertices, where harder ones are those with higher density of edges. In addition, the presence of twin vertices makes instances easier to solve, specially when $\mathscr{F}_2$ is chosen.

Besides this tool has been very useful for checking our theoretical results, we have tested the conjecture over all connected graphs up to 10 vertices (about 12 million graphs). These instances are provided by Brendan McKay:

`http://users.cecs.anu.edu.au/~bdm/data/graphs.html`

while a DSATUR code by Rhyd Lewis have been used for obtaining $\chi(G)$:

`http://rhydlewis.eu/resources/gCol.zip`

For each instance, we assign its chromatic number to UB and solve $\mathscr{F}_2$ (indeed, it is not necessary to reach optimality; the solver is interrupted

Vertices	Density	Edges	Time	Edges	Time	Edges	Time
20	Low	55	0.03	40	0.02	49	0.02
	Med.	88	0.09	96	0.17	80	0.08
	High	125	14.1	145	2.48	138	3.47
25	Low	67	0.03	83	0.11	76	0.03
	Med.	161	332	148	8.32	168	896
	High	237	54.3	216	198	231	7.09
30	Low	130	0.20	113	0.05	104	1.31
	Med.	223	1696	213	666	219	53.5
	High	327	–	316	–	313	733
30+20f		503	0.12(0.23)	573	0.23(0.39)	559	0.19(0.44)
30+10t+10f		668	0.61(0.84)	738	1.06(6.45)	724	0.52(12.4)
30+20t		713	1.56(4.92)	783	2.38(7.05)	769	3.78(5.63)

Table 1: Time in seconds needed to solve random instances

when a feasible solution is found). The test fails if an infeasible model is reached, which means that a counterexample to the conjecture is found. Fortunately, it finished with success (i.e. the conjecture is valid for the set of graphs tested). In particular, all graphs of 9 vertices (261080) were solved in 285 seconds and all graphs of 10 vertices (11716571) were solved in 25027 seconds. The average of time elapsed for each instance of 9 and 10 vertices is 1.09 and 2.14 milliseconds respectively.

Closed additive colorings. Because of Additive Coloring Conjecture, it is natural to wonder if a similar conjecture can be proposed by considering a local identification problem where closed neighborhoods are used instead of open ones. More specifically, consider a graph G without true twins and call *closed additive k-coloring* to a labeling $f : V \to [k]$ such that $f(N[u]) \neq f(N[v])$ for all edges $(u, v) \in E$. Then, denote with $\eta[G]$ the least number k for which G has a closed additive k-coloring. We wonder whether the inequality $\eta[G] \leq \chi(G)$ holds for all graph G without true twins, since there are several cases where this happens (for instance, $\eta[G] = 1$ if and only if $\eta(G) = 1$, then $\eta[G] \leq \chi(G)$ for graphs G such that $\eta[G] = 1$). We have seen empirically that the inequality always holds for graphs up to 10 vertices. Unfortunately, we found a family of counterexamples such that $\eta[G] - \chi(G)$ can be as large as one wishes. Indeed, let K_n be a complete graph of n vertices, with $n \geq 4$. Construct a graph G by replacing every edge (u, v) of K_n by a path $\{u, w^1_{uv}, w^2_{uv}, v\}$. The assignment $f(v) = 3$ for all $v \in V(K_n)$, and $f(w^i_{uv}) = i$ for all $(u, v) \in E(K_n)$ is a 3-coloring of G. In fact, $\chi(G) = 3$ since G contains a C_9. Now, if f' is a closed additive k-coloring of G, then $f'(N[w^1_{uv}]) \neq f'(N[w^2_{uv}])$ for all $(u, v) \in E(K_n)$. Hence, $f'(u) \neq f'(v)$ and each vertex from $V(K_n)$ must have a different labeling. Therefore, $\eta[G] \geq n$. Figure 2 displays the smallest counterexample generated with this construction, which has 16 vertices.

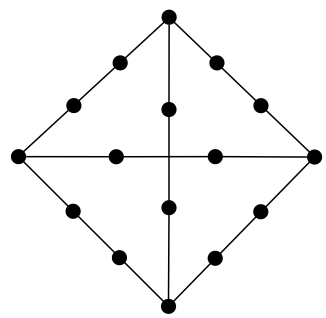

Figure 2: Graph G such that $\eta[G] = 4 > 3 = \chi(G)$

A formulation $\mathscr{F}'$ that computes $\eta[G]$ is obtained by replacing constraints $f(N(u)) - f(N(v)) + M_{uv}z(u, v) \leq M_{uv} - 1$ in $\mathscr{F}$ by $f(N[u]) - f(N[v]) + M_{uv}z(u, v) \leq M_{uv} - 1$, where $M_{uv} = 1 + |N[u] \backslash N[v]|UB - |N[v] \backslash N[u]|$ for all $(u, v) \in E_2$. We have also run the experiment with formulation $\mathscr{F}'$ in order to check if $\eta[G] \leq \chi(G)$ for all connected graphs up to 10 vertices and without true twins. Again, the test finished with success although it took more time, probably because no additional valid inequalities have been added to $\mathscr{F}'$ (as in the case of $\mathscr{F}$ and $\mathscr{F}_2$). In particular, all graphs of 9 vertices (197772) were solved in 724 seconds and all graphs of 10 vertices (9721362) were solved in 46062 seconds. The average of time elapsed for each instance of 9 and 10 vertices is 3.66 and 4.74 milliseconds respectively.

About this parameter, $\eta[G]$, M. Axenovich, J. Harant, J. Przybyło, R. Soták, M. Voigt and J. Weidelich published *A note on adjacent vertex distinguishing colorings of graphs* (Discr. Appl. Math. **205** (2016) 1–7) where they show that $\eta[G] \leq \Delta^2 - \Delta + 1$ (the same bound of Akbari et al. for $\eta(G)$ [2] and another family of counterexamples for the conjecture among other results concerning this parameter. They also extend the definition for the cases where G has true twins.

C Examples of graphs and its optimal additive colorings

Figure 3 presents examples of a 4-fan, a windmill graph, a wheel, a complete sun, a cycle sun and a headless thick spider of order 5. In the drawing of the thick spider, the edges (u_j, v_k) with $j \neq k$ have been removed for the sake of clarity. Instead, a dashed line connecting u_j and v_j have been added to remember that these vertices are not connected.

Also additive colorings are shown on Figure 4: on the left column, the labelings of an optimal additive coloring are displayed, while on the right, the values of $f(N(v))$ for each vertex v are reported.

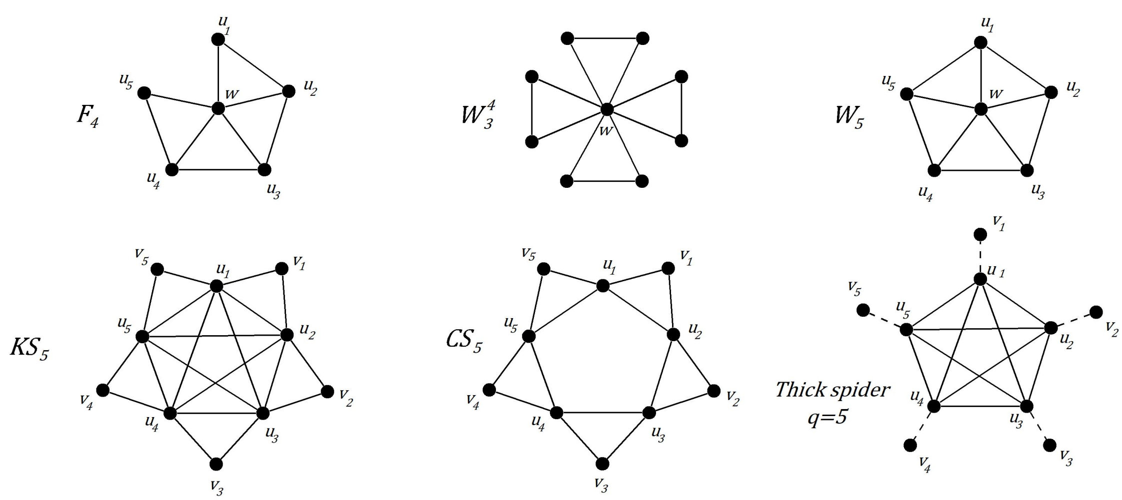

Figure 3: Examples of graphs

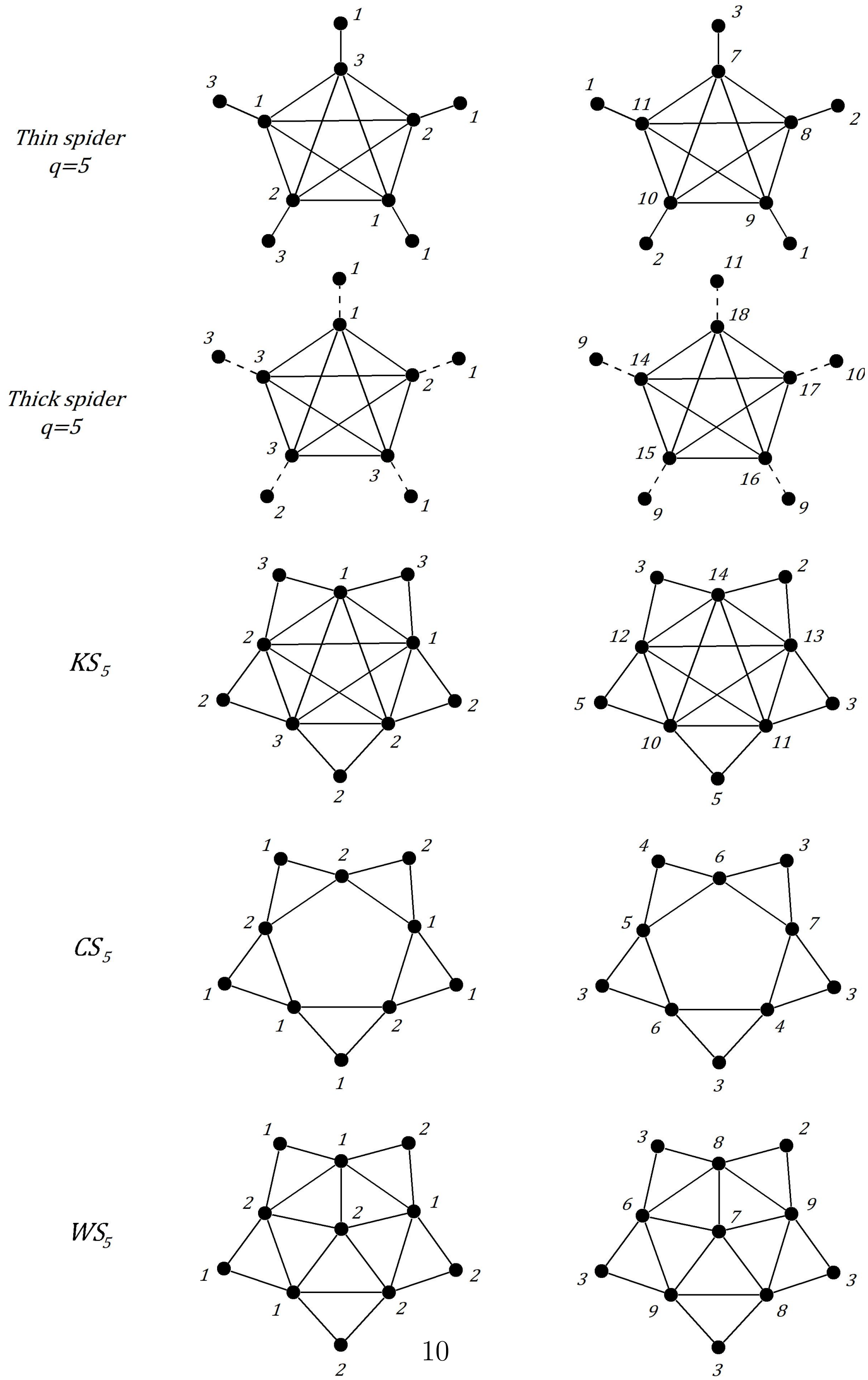

Figure 4: Additive colorings of spiders and suns